SHOCK STRUCTURE IN A VISCOELASTIC MEDIUM WITH A NONLINEAR DEPENDENCE OF THE MAXWELLIAN VISCOSITY ON THE PARAMETERS OF THE MATERIAL

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The shock-profile structure in a viscoelastic medium with a nonlinear dependence of the Maxwellian viscosity χ (a quantity inverse to the tangential stress-relaxation time τ) on the substance parameters is investigated in this paper on the basis of a model proposed in [1]. The presence of such a dependence of the relaxation time permitted the extraction of sections with an abrupt change in the quantities on the profile, called plastic waves, by using additional relationships. The model of the isotropic medium used is characterized by an equation of state in the form of a dependence of the density of the internal energy E per unit mass on the strain-tensor invariants and the entropy S.

1. ONE-DIMENSIONAL SYSTEM OF EQUATIONS

Maxwell viscoelastic terms which describe the process of tangential stress deviator relaxing with time are included in the equations of the medium [1] to describe the plastic-deformation processes.

Plastic deformations proceed as the entropy of the material increases. The characteristic time τ of the relaxation process can hence vary between broad limits as a function of the state of the medium; its temperature, degree of compression, and intensity of the tangential stresses. Condensed substances, metals, powders, liquids, should be among such media.

As has been mentioned in [2], a metal under normal conditions has a characteristic time τ on the order of several hours, while τ drops to 10^{-5} sec under shock loadings [3].

In this connection, an investigation of the shock structure in viscoelastic media with strongly varying relaxation time of the tangential stresses is of considerable interest. The form of the equations of state proposed in [4], and interpolation formulas for the dependence of the magnitude of the Maxwellian viscosity on the temperature, compression, and tangential stress intensities presented in [3] were used in computations of specific examples.

Let us note that the temperature dependence of the flow stress had to be taken into account in [5, 6] devoted to shocks in Plexiglas, which were studied experimentally by using an elastoplastic scheme.

Let us examine the system of differential equations describing the motion of a viscoelastic medium parallel to the selected x axis in the (x, y, z) space

$$\partial\rho / \partial t + \partial\rho u / \partial x = 0 \tag{1.1}$$

$$\partial\rho u / \partial t + \frac{\partial [\rho u^2 - \sigma_1]}{\partial x} = 0 \tag{1.2}$$

$$\frac{\partial\rho (E + u^2/2)}{\partial t} + \frac{\partial [\rho u (E + u^2/2) - \sigma_1 u]}{\partial x} = 0 \tag{1.3}$$

$$\frac{\partial\beta}{\partial t} + u\frac{\partial\beta}{\partial x} = -\tau^{-1}\left(\beta - \frac{\alpha + \beta + \gamma}{3}\right) \tag{1.4}$$

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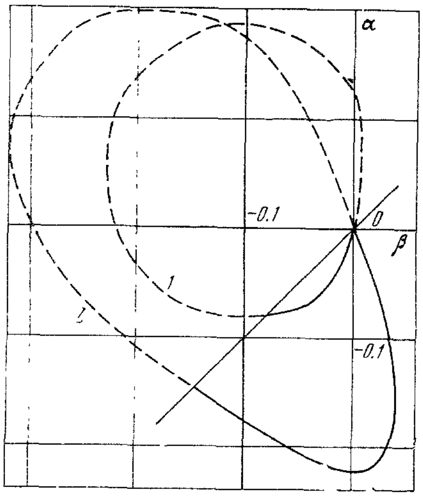

Fig. 1

Here x and t are the space coordinate and the time, u is the velocity of substance motion along the x axis, the quantities α, β and γ are the logarithms of the relative elongations k_1, k_2, k_3 along the x, y, z axes: $\alpha = \ln k_1$, $\beta = \ln k_2$, $\gamma = \ln k_3$.

Since the medium is considered isotropic, then $\beta \equiv \gamma$ during the whole motion process. The quantity ρ is the density, it being related to α, β and γ by the relationship

$$\rho = \rho^\circ e^{-(\alpha+\beta+\gamma)} \tag{1.5}$$

where ρ° is the density of the substance in the initial state.

The density of the substance internal energy per unit mass is related to the density of the entropy S per unit mass and the quantities α, β and γ by the equation of state for an isotropic medium

$$E = E(\alpha, \beta, \gamma, S) \tag{1.6}$$

Here E is a symmetric function of α, β, γ. The quantity σ_1 is the principal stress directed along the x axis. Because of the isotropy of the medium, it can be assumed that the principal stresses σ_2 and σ_3 are directed along the y and z axes. In this case the stresses σ_i are related to the strains by the formulas

$$\sigma_1 = \rho \frac{\partial E}{\partial \alpha}, \quad \sigma_2 = \rho \frac{\partial E}{\partial \beta}, \quad \sigma_3 = \rho \frac{\partial E}{\partial \gamma} \tag{1.7}$$

Because of isotropy $\sigma_2 \equiv \sigma_3$.

The relaxation time $\tau > 0$ is a function of the state of the medium, i.e.,

$$\tau = \tau(\alpha, \beta, \gamma, S) \tag{1.8}$$

The system (1.1)-(1.8) is a one-dimensional version of the system of equations proposed in [1], referred to the principal axes of the stress tensor. The system (1.1)-(1.8) and the system in [1] differ by the form of the members in the right side of (1.4), which describe the plastic strain process. The use of a phenomenological approach yields no advantages whatever over these methods of introducing the relaxation terms.

Let us present the equation for the entropy which is a corollary of (1.1)-(1.7):

$$\rho E_S \left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x}\right) = \frac{2}{3} \frac{(\alpha - \beta)(\sigma_1 - \sigma_2)}{\tau} \tag{1.9}$$

as well as formulas for the speeds of sound: longitudinal,

$$c = (\partial^2 E / \partial \alpha^2 - \partial E / \partial \alpha)^{1/2} \tag{1.10}$$

and transverse,

$$b = ({}^1/_2 \partial E / \partial D)^{1/2} \tag{1.11}$$

$$D = \frac{1}{2}\left[\left(\alpha - \frac{\alpha+\beta+\gamma}{3}\right)^2 + \left(\beta - \frac{\alpha+\beta+\gamma}{3}\right)^2 + \left(\gamma - \frac{\alpha+\beta+\gamma}{3}\right)^2\right] \tag{1.12}$$

2. RELATIONSHIPS ON THE SHOCKS

Let us call a shock a solution of the system (1.1)-(1.9) of the form

$$\alpha = \alpha(x - Ut), \quad \beta \equiv \gamma \equiv \beta(x - Ut), \quad S = S(x - Ut) \tag{2.1}$$

Let the shock be at rest in the selected coordinate system, i.e., U = 0. Then (2.1) satisfies the system

$$\begin{cases} d\rho u / dx = 0, & d(\rho u^2 - \sigma_1) / dx = 0 \\ \dfrac{d[\rho u (E + u^2/2) - \sigma_1 u]}{dx} = 0, & u \dfrac{d\beta}{dx} = \dfrac{\alpha - \beta}{3\tau} \end{cases} \tag{2.2}$$

At the ends of the shock (for $x \to \pm\infty$) the quantities α, β and S should take on finite values, and $d\alpha/dx$, $d\beta/dx$ and dS/dx vanish. It follows from (2.2) that $\alpha = \beta = \gamma = {}^1/_3 \ln(\rho^\circ / \rho)$ and $\sigma_1 = \sigma_2 = \sigma_3 = -p$ at the ends

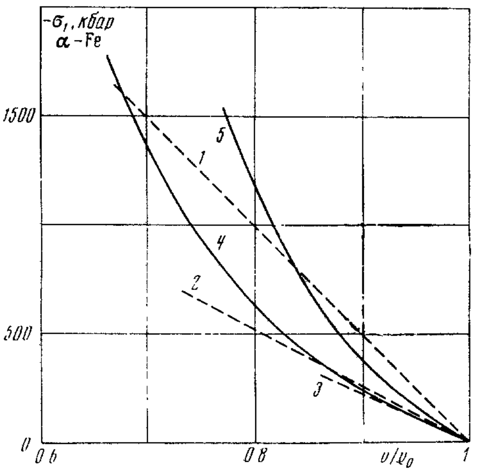

Fig. 2

of the shock, i.e., the medium should be subjected to hydrostatic pressure. Letting w denote the stream of substance through the shock,

$$w = \rho u \tag{2.3}$$

we obtain by using (2.2) that the values of the quantities at the ends of the shock are connected by relationships analogous to the gasdynamic relationships

$$\begin{cases} [p] = w^2 [1/\rho], \quad [u] = -w[1/\rho] \\ [E] + \frac{p_0 + p_1}{2}[1/\rho] = 0 \\ \alpha = \beta = \gamma = {}^1/_3 \ln(\rho^\circ/\rho) \end{cases} \tag{2.4}$$

Here p_0 and p_1 are the magnitudes of the pressure in front of and behind the wave. As follows from (2.2) and (2.3), the thermodynamic quantities and velocity within the wave are connected by the relationships

$$u - u_0 = w(1/\rho - 1/\rho_0) \tag{2.5}$$

$$\sigma_1 + p_0 = w^2(1/\rho - 1/\rho_0) \tag{2.6}$$

$$E - E_0 + [(p_0 - \sigma_1)/2](1/\rho - 1/\rho_0) = 0 \tag{2.7}$$

$$w\frac{d\beta}{dx} = \frac{\rho}{3}\frac{\alpha - \beta}{\tau} \tag{2.8}$$

Here $\beta = \gamma$, u_0, ρ_0, p_0, E_0 are the magnitudes of the velocity, density, pressure, and energy ahead of the wave. The problem of constructing the wave profile reduces to integrating the system (2.6)-(2.8) with the additional relation (2.5), and the problem of determining the state behind the wave in terms of the velocity w, and the state ahead of the wave ρ_0, S_0 reduces to the solution of the system (2.4). The problem of determining the state behind the wave is solved in complete analogy with gasdynamics: if the equation of state (1.6) satisfies the inequalities

$$\begin{gathered} \partial^2 E/\partial\rho^2 > 0, \quad \partial^2 E/\partial\rho\partial S > 0, \quad \partial^2 E/\partial S^2 > 0, \\ \partial^3 E/\partial\rho^3 < 0, \quad \partial E/\partial D > 0 \end{gathered} \tag{2.9}$$

for $\alpha = \beta = \gamma$, then the Hugoniot adiabat in the $(p, 1/\rho)$ plane,

$$E_1 - E_0 + \frac{p_1 + p_0}{2}(1/\rho_1 - 1/\rho_0) = 0, \quad \alpha = \beta = \gamma \tag{2.10}$$

has only two intersections with the Michelson line

$$[p] = -w^2[1/\rho], \quad \alpha = \beta = \gamma \tag{2.11}$$

corresponding to the initial and final state $(p_0, 1/\rho_0)$ and $(p_1, 1/\rho_1)$.

The Hugoniot adiabat in the initial state is tangent to a line with the slope $w = \rho_0(c_0^2 - 4/3b_0^2)$ which is the modulus of volume compression. The secant with the slope $w_h = \rho_0 c_0$ corresponds to the adiabat of the so-called "Hugoniot point" $(p_h, 1/\rho_h)$. Supersonic wave velocities correspond to points of the Hugoniot adiabat lying above $(p_h, 1/\rho_h)$, and subsonic to the rest. Presented below are values of $(p_h, \rho_h/\rho^\circ)$ for some metals.

Metal	Fe	Al	Cu	Ni	Pb	Ti
p_h, kbar	386.4	112.1	207.8	227.5	36.00	115.8
ρ_h/ρ°	1.175	1.116	1.117	1.090	1.069	1.082

3. SHOCK STRUCTURE

Let us consider the structure of a shock profile. It follows from (2.6)-(2.8) that the problem of constructing the profile reduces to solving the system of equations

$$p_0 + \sigma_1 = w^2(1/\rho - 1/\rho_0) \tag{3.1}$$

$$E - E_0 + \frac{p_0 - \sigma_1}{2}(1/\rho - 1/\rho_0) = 0 \tag{3.2}$$

under the condition $\beta = \gamma$ and the quadrature

$$dx / d\beta = 3\tau w / \rho (\alpha - \beta) \tag{3.3}$$

The system (3.1), (3.2) determines a curve in the space α, β, S which we call the curve of possible states. The curve of possible states is projected on the α, β plane as a closed convex curve intersecting the line $\alpha = \beta$ and two, and only two, points corresponding to the beginning and ending state of the shock. Curve 1 in Fig. 1 corresponds to $|w| < \rho_0 c_0$ and curve 2, to $|w| > \rho_0 c_0$. Here (α-phase) iron was taken as the material. Corresponding to curve 2 is $|w|/\rho_0 c_0 = 1.48$ and to curve 1 is $|w| \rho_0 c_0 = 0.88$. For a subsonic wave ($|w| < \rho_0 c_0$) the portion of the curve of (3.1), (3.2) in the half-plane $\beta > \alpha$ and the quadrature (3.3) yield the solution of the problem of constructing the profile. At supersonic wave velocities in the neighborhood of the initial state, we find from (3.1) and (3.2)

$$\frac{\alpha - \alpha_0}{\beta - \beta_0} = -2 \frac{c_0^2 - 2b_0^2 - w^2 / \rho_0^2}{c_0^2 - w^2 / \rho_0^2} \tag{3.4}$$

For supersonic waves, $(\alpha - \alpha_0)/(\beta - \beta_0) < 0$. It hence follows that there should be a discontinuity in the wave profile. From (3.3) and $w(d\beta/dx) < 0$ there follows $\beta > \alpha$; the profile continuity would contradict the monotonicity of $\beta(x)$ because of (3.4).

In this connection, the problem of constructing a generalized (discontinuous) solution for the case $|w| > \rho_0 c_0$ must be solved.

Let us introduce a discontinuity, an elastic predecessor, into the solution by imposing the additional relationship $[\beta] = 0$ thereon. This equality assures the absence of stress relaxation within the elastic jump. The introduction of the discontinuity $[\beta] = 0$ in the wave profile can be performed uniquely, namely, it is at the beginning of the wave, emergence from the initial state is impossible by using a smooth solution, and a curve of possible states has a single point of intersection with the line $\beta = \beta_0$ (see Fig. 1) in the $\beta > \alpha$ plane because of the convexity. A smooth passage to the position corresponding to the end of the shock from this point is possible along the curve of possible states in the plane $\beta > \alpha$.

For $|w| > \rho_0 c_0$ the shock is comprised of the jump governed by the relations (3.1), (3.2) and $\beta = \beta_0$, which we call the elastic wave, and a smooth section governed by the relations (3.1)-(3.3), which can be called the relaxation layer. There is no elastic jump in the case $|w| < \rho_0 c_0$. In both cases, the relationship $\beta > \alpha$, i.e., $\sigma_2 > \sigma_1$, is satisfied everywhere on the shock. It is convenient to represent the shock in the $(-\sigma_1, 1/\rho)$ plane. Shown in Fig. 2 are the curves 4 and 5 of the shock adiabats corresponding to the final state and the elastic jump. Lines 1, 2, and 3 are the transition lines of (3.1). The elastic adiabat 5 in Fig. 2 is tangent to the transition line 2 at the initial point, corresponding to the Hugoniot point of the adiabat of the final state. This means that the elastic jump diminishes as the shock attenuates and vanishes for $|w| < \rho_0 c_0$.

For equations of state satisfying the inequalities (2.9) for $\beta = \text{const}$, the elastic jump satisfies the requirements of evolutionarity.

4. PLASTIC WAVES

The question of the shock structure was examined in Sec. 3 without taking account of the possible singularities associated with the dependence of the relaxation time τ on the parameters of the medium. If τ were constant, then the relaxation layer would represent a smooth transition. For real substances the relaxation time depends essentially on the temperature T, the density ρ, and the tangential stress intensity σ. For metals (see [3])

$$\tau = \tau_0 (\sigma/\sigma_0)^m e^{-U(\sigma, T)/RT} \tag{4.1}$$

Here $U(\sigma, T)$ is the activation energy, R is the universal gas constant, and τ_0, σ_0 are constants. Such acute dependences for the relaxation time for metals result in a section with a steep front being formed on the relaxation layer, which can be isolated in an individual wave.

Such a front structure, split into the elastic predecessor and a plastic wave, has been observed repeatedly in experimental investigations of shocks in metals and has been described qualitatively in [7], for instance.

In the terminology used in the paper, this description can be made as follows: Let us be given the characteristic duration Δt of the plastic wave, and let us construct a curve governed by the relationships (3.1), (3.2), and $\tau = \Delta t$ [τ from (4.1)] in the plane $(-\sigma_1, 1/\rho)$ for different values of the velocity w.

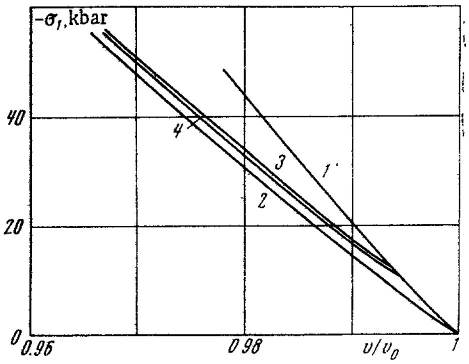

Fig. 3

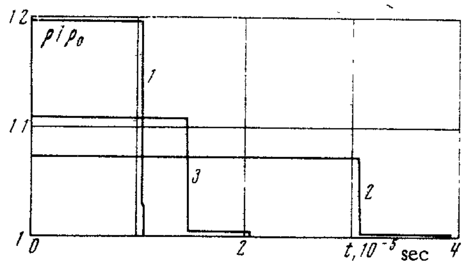

Fig. 4

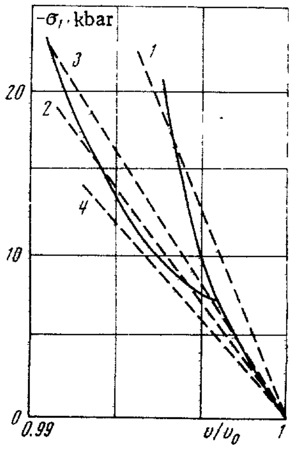

Fig. 5

In addition to the elastic and hydrodynamic adiabat curves 1 and 2 in Fig. 3, the curves 3 and 4 are presented which correspond to the distinct values $\tau = 1$ μsec and 100 μsec (copper was selected as material). We call the curves corresponding to these values the plastic adiabats. Let us replace the plastic waves of width $\approx \Delta t$ by discontinuous solutions for which the magnitudes of the jumps are calculated by using the plastic adiabats.

Let us call this process the extraction of the plastic waves.

Let us consider diverse cases of extracting plastic waves in the front structure by the example of copper. Analogous reasoning can be carried out for other metals also. They depend on the shock velocities. For strong shocks $|w| > \rho_0 c_0$ (the pressure behind the front is greater than 200 kbar), the plastic adiabats are practically indistinguishable from the hydrodynamic adiabats, and the conditions

$$\tau = \Delta t_1, \quad \tau = \Delta t_2$$

are substantially equivalent to the conditions $\alpha = \beta$. This means that for sufficiently strong shocks in metals, the distinction between σ_1 and σ_2 can be neglected, and the hydrodynamic approach can be used, as is often done. From this viewpoint it is understandable why a metal behaves as a fluid at high pressures. Presented in Fig. 4 is the shock profile 1 corresponding to the secant 1 (Fig. 2, curve 5). It is seen from the front structure that in this case the elastic jumps at the thicknesses Δt_1 and Δt_2 are indistinguishable from the relaxation layer, and, hence, the plastic wave can be isolated by the fundamental relationships

$$E_1 - E_0 - \frac{\sigma_{10} + \sigma_{11}}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) = 0, \quad \sigma_{11} - \sigma_{10} = w^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) \tag{4.2}$$

$$u_1 - u_0 = \pm\sqrt{(\sigma_{11} - \sigma_{10})(1/\rho_1 - 1/\rho_0)} \tag{4.3}$$

and by one additional relationship behind the shock,

$$\tau(\alpha_1, \beta_1, S_1) = \Delta t \tag{4.4}$$

which corresponds to the intersection of the plastic adiabat with the secant 1 in Fig. 2. The Euler velocity D of the boundary is determined by the relationship

$$D = \frac{\sqrt{\rho_1}u_1 - \sqrt{\rho_2}u_2}{\sqrt{\rho_1} - \sqrt{\rho_0}} \pm \sqrt{\frac{\sigma_{11} - \sigma_{10}}{\rho_0 - \rho_1}} \tag{4.5}$$

For waves at a subsonic velocity ($|w| < \rho_0 c_0$, secant 2 in Fig. 2, curve 5), the characteristic profile is represented by curve 2 in Fig. 4. Since the wave proceeds at a subsonic velocity, the isolation of the wave is accomplished by means of two additional relationships, in addition to the fundamental (4.2) and (4.3): in front of the wave,

$$\tau(\alpha_0, \beta_0, S_0) = \Delta t \tag{4.6}$$

and behind the wave (4.4), which corresponds to the intersection between the plastic adiabat and the secant 2 in Fig. 5 at two points.*

In the intermediate case between those described above, when shock-wave propagation is at a velocity somewhat exceeding the speed of sound (secant 3 in Fig. 5), the front structure has the shape of the curve shown in Fig. 4 by curve 3. In this case there is the elastic jump advancing by the plastic wave. In this case the plastic wave is isolated by using the two additional relationships (4.4) and (4.6), as in the

* Adjustment of $\tau(\alpha, \beta, S)$ to the same constant Δt is not necessary in (4.4) and (4.6).

subsonic mode, which also corresponds to the intersection between the secant 3 in Fig. 5 and the plastic adiabat at two points. The need to use the two additional relationships to isolate the plastic wave is related to the fact that the velocity of the plastic wave is less than the speed of sound ahead of the wave. This fact follows from the circumstance that the speed of sound behind the elastic jump grows along the wave profile.

Plastic waves of width greater than Δt were not considered discontinuous. Secants of the shape 4 in Fig. 5 correspond to them.

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